

Uniform resolvent convergence for strip with fast oscillating boundary

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Abstract

In a planar infinite strip with a fast oscillating boundary we consider an elliptic operator assuming that both the period and the amplitude of the oscillations are small. On the oscillating boundary we impose Dirichlet, Neumann or Robin boundary condition. In all cases we describe the homogenized operator, establish the uniform resolvent convergence of the perturbed resolvent to the homogenized one, and prove the estimates for the rate of convergence. These results are obtained as the order of the amplitude of the oscillations is less, equal or greater than that of the period. It is shown that under the homogenization the type of the boundary condition can change.

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Introduction

There are many papers devoted to the homogenization of boundary value problems in domains with fast oscillating boundary. The simplest example of such boundary is given by the graph of the function $x_2 = \eta(\varepsilon)b(x_1\varepsilon^{-1})$, where ε is a small positive parameter, $\eta(\varepsilon)$ is a positive function tending to zero as $\varepsilon \rightarrow +0$, and b is a smooth periodic function. The parameter ε describes the period of the boundary oscillations while $\eta(\varepsilon)$ is their amplitude.

Most of the papers on such topic are devoted to the case of bounded domains with fast oscillating boundary. Not trying to cite all papers in this field, we just mention [1, Ch. III, Sec. 4], [2]–[23], see also the references therein. Main results concerned the identification of the homogenized problems and proving the convergence theorems for the solutions. The homogenized (limiting) problems were the boundary value problems for the same equations in the same domains but with the mollified boundary instead of the oscillating one. The type of the condition on the mollified boundary depended on the original boundary condition and the geometry of the oscillations. If the amplitude of the oscillations is of the same order as the period (i.e., in above example $\eta \sim \varepsilon$),

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the homogenized boundary condition is of the same type as the original condition on the oscillating boundary. In the case of Robin condition the homogenization gives rise to an additional term in the coefficient in the homogenized boundary condition; this term reflected the local geometry of the boundary oscillations. If the period of the boundary oscillations is smaller (in order) than the amplitude, the boundary is highly oscillating. To authors knowledge, such case was considered in [14], [15], and [18]. In [15] the model was the spectral problem for the biharmonic operator with Dirichlet condition, while in [14] the Robin problem for Poisson equation was studied. In the former case in particular it was shown that the homogenized boundary condition was the Dirichlet one while in the latter the authors discovered that in the case of highly oscillating boundary the homogenized boundary condition is also the Dirichlet while the perturbed problem involved the Robin condition. In [18] the Navier-Stokes system was considered and the boundary condition on the oscillating boundary was the Robin one with the small parameter involved in the coefficients. The homogenized problems were found and the weak convergence was established.

In [9]–[12] a boundary value problem for the semilinear elliptic equation with non-linear boundary conditions of Robin type in a bounded domain was considered. The domain was assumed to be perturbed and the only assumption was that the perturbed domain converged to a certain limiting one in the sense of Hausdorff and the same was valid for the boundaries. This setting includes also the case of fast oscillating boundary. The main result was the proof of the convergence of the perturbed solution to the limiting one in W_2^1 -norm and similar statements for the spectra.

It should be said that there are also many papers devoted to the problems in the domains with the oscillating boundaries when the period of the oscillations is small and the amplitude is finite. Since in our case the amplitude is small, it is quite a different problem. This is why here we do not dwell on the problems with finite amplitude.

Most of the results on the convergence of the solutions were established in the sense of the weak or strong resolvent convergence, and the resolvents were also treated in various possible norms. In some cases the estimates for the convergence rate were proven. It was also shown that constructing the next terms of the asymptotics for the perturbed solutions one get the estimates for the convergence rate or improves it [3], [4], [13], [15], [16], [20], [21], [23]. In some cases complete asymptotic expansions were constructed [5], [8], [22], [24].

One more type of the established results is the uniform resolvent convergence for the problems. Such convergence was established just for few models, see [1, Ch. III, Sec. 4], [23]. The estimates for the rates of convergence were also established. In both papers the amplitude and the period of oscillations were of the same order. At the same time, the uniform resolvent convergence for the models considered in the homogenization theory is a quite strong results. Moreover, recently the series of papers by M.Sh. Birman, T.A. Suslina and V.V. Zhikov, S.E. Pastukhova have stimulated the interest to this aspect, see [25]–[38], the references therein and further papers by these authors. It was shown that the uniform resolvent convergence holds true for the elliptic operators with fast oscillating coefficients and the estimates for the rates of convergence were obtained. There are also same results for some problems in bounded domains, see [37]. Similar results but for the boundary homogenization were established in [39]–[43]. Here the Laplacian in a planar straight infinite strip with frequently alternating boundary conditions was considered. Such boundary conditions were imposed by partitioning the boundary into small segments where Dirichlet and Robin conditions were imposed in turns. The homogenized problem involves one of the classical boundary conditions instead of the alternating ones. For all possible homogenized problems the uniform resolvent and the estimates for the rates of convergence were proven and the asymptotics for the spectra were constructed.

In the present paper we also consider the boundary homogenization for the elliptic operators in unbounded domains but the perturbation is a fast oscillating boundary. As the domain we choose a planar straight infinite strip with a periodic fast oscillating boundary; the operator is a general self-adjoint second order elliptic operator. The op-

erator is regarded as an unbounded one in an appropriate L_2 space. On the oscillating boundary we impose Dirichlet, Neumann, or Robin condition. Apart from a mathematical interest to this problem, as a physical motivation we can mention a model of a planar quantum or acoustic waveguide with a fast oscillating boundary.

Our main result is the form of the homogenized operator and the uniform resolvent convergence of the perturbed operator to the homogenized one. This convergence is established in the sense of the norm of the operator acting from L_2 into W_2^1 . The estimates for the rate of convergence are provided. Most of the estimates are sharp. We show that in the case of the Dirichlet or Neumann condition on the oscillating boundary the homogenized problem involves the same condition on the mollified boundary no matter how the period and amplitude of the oscillations behave. Provided the amplitude is not greater than the period (in order), the Robin condition on the oscillating boundary leads us to a similar condition but with an additional term in the coefficient. If the amplitude is greater than the period, the homogenization transforms the Robin condition into the Dirichlet one. The last result is in a good accordance with a similar case treated in [14]. The difference is that in [14] the strong resolvent convergence was proven provided the coefficient in the Robin condition is positive, while we succeeded to prove the uniform resolvent convergence provided the coefficient is either positive or non-negative and vanishing on the set of zero measure.

1 The problem and the main results

Let $x = (x_1, x_2)$ be the Cartesian coordinates in \mathbb{R}^2 , ε be a small positive parameter, $\eta = \eta(\varepsilon)$ be a non-negative function uniformly bounded for sufficiently small ε , $b = b(t)$ be a non-negative 1-periodic function belonging to $C^2(\mathbb{R})$. We define two domains, cf. fig. 1,

$$\Omega_0 := \{x : 0 < x_2 < d\}, \quad \Omega_\varepsilon := \{x : \eta(\varepsilon)b(x_1\varepsilon^{-1}) < x_2 < d\},$$

where $d > 0$ is a constant, and its boundaries are indicated as

$$\Gamma := \{x : x_2 = d\}, \quad \Gamma_0 := \{x : x_2 = 0\}, \quad \Gamma_\varepsilon := \{x : x_2 = \eta(\varepsilon)b(x_1\varepsilon^{-1})\}.$$

By $A_{ij} = A_{ij}(x)$, $A_j = A_j(x)$, $A_0 = A_0(x)$, $i, j = 1, 2$, we denote the functions defined on Ω_0 and satisfying the belongings $A_{ij} \in W_\infty^2(\Omega_0)$, $A_j \in W_\infty^1(\Omega_0)$, $A_0 \in L_\infty(\Omega_0)$. Functions A_{ij} , A_j are assumed to be complex-valued, while A_0 is real-valued. In addition, functions A_{ij} satisfy the ellipticity condition

$$A_{ij} = \overline{A_{ji}}, \quad \sum_{i,j=1}^2 A_{ij} z_i \overline{z_j} \geq c_0(|z_1|^2 + |z_2|^2), \quad x \in \Omega_0, \quad z_j \in \mathbb{C}. \quad (1.1)$$

By $a = a(x)$ we denote a real function defined on $\{x : 0 < x_2 < \delta\}$ for some small fixed δ , and it is supposed that $a \in W_\infty^1(\{x : 0 < x_2 < \delta\})$.

The main object of our study is the operator

$$-\sum_{i,j=1}^2 \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum_{j=1}^2 A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 \quad \text{in } L_2(\Omega_\varepsilon) \quad (1.2)$$

subject to Dirichlet condition on Γ . On the other boundary we choose either Dirichlet condition

$$u = 0 \quad \text{on } \Gamma_\varepsilon,$$

or Robin condition

$$\left(\frac{\partial}{\partial \nu^\varepsilon} + a \right) u = 0 \quad \text{on } \Gamma_\varepsilon, \quad \frac{\partial}{\partial \nu^\varepsilon} = - \sum_{i,j=1}^2 A_{ij} \nu_j^\varepsilon \frac{\partial}{\partial x_i} - \sum_{j=1}^2 \overline{A_j} \nu_j^\varepsilon,$$

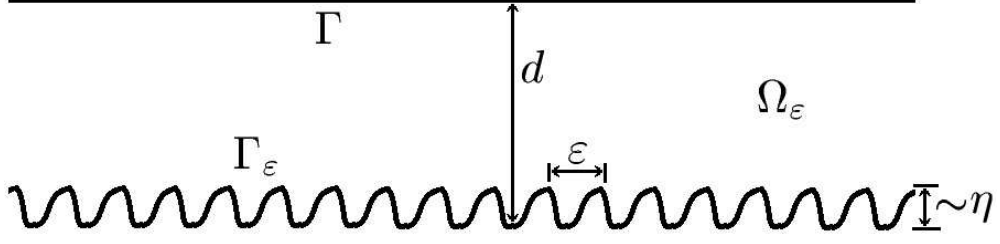


Figure 1: Domain with oscillating boundary

where $\nu^\varepsilon = (\nu_1^\varepsilon, \nu_2^\varepsilon)$ is the outward normal to Γ_ε . In the case of Dirichlet condition on Γ_ε we denote this operator as $\mathcal{H}_{\varepsilon,\eta}^D$, while for Robin condition it is $\mathcal{H}_{\varepsilon,\eta}^R$. The former includes also the case of Neumann condition since the function a can be identically zero.

Rigorously we introduce $\mathcal{H}_{\varepsilon,\eta}^D$ as the lower-semibounded self-adjoint operator in $L_2(\Omega_\varepsilon)$ associated with the closed symmetric lower-semibounded sesquilinear form

$$\begin{aligned} \mathfrak{h}_{\varepsilon,\eta}^D(u, v) := & \sum_{i,j=1}^2 \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^2 \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_\varepsilon)} \\ & + \sum_{j=1}^2 \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_\varepsilon)} + (A_0 u, v)_{L_2(\Omega_\varepsilon)} \end{aligned}$$

in $L_2(\Omega_\varepsilon)$ with the domain $\mathfrak{D}(\mathfrak{h}_{\varepsilon,\eta}^D) := W_{2,0}^1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$. Hereinafter $\mathfrak{D}(\cdot)$ is the domain of a form or an operator, and $W_{2,0}^j(\Omega, S)$ denotes the Sobolev space consisting of the functions in $W_2^j(\Omega)$ with zero trace on a curve S lying in a domain $\Omega \subset \mathbb{R}^2$. The operator $\mathcal{H}_{\varepsilon,\eta}^R$ is introduced in the same way via the sesquilinear form

$$\begin{aligned} \mathfrak{h}_{\varepsilon,\eta}^R(u, v) := & \sum_{i,j=1}^2 \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^2 \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_\varepsilon)} \\ & + \sum_{j=1}^2 \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_\varepsilon)} + (A_0 u, v)_{L_2(\Omega_\varepsilon)} + (au, v)_{L_2(\Gamma_\varepsilon)} \end{aligned}$$

with the domain $\mathfrak{D}(\mathfrak{h}_{\varepsilon,\eta}^R) := W_{2,0}^1(\Omega_\varepsilon, \Gamma)$.

The main aim of the paper is to study the asymptotic behavior of the resolvents of $\mathcal{H}_{\varepsilon,\eta}^D$ and $\mathcal{H}_{\varepsilon,\eta}^R$ as $\varepsilon \rightarrow +0$. To formulate the main results we first introduce some additional operators.

By \mathcal{H}_0^D we denote operator (1.2) in $L_2(\Omega_0)$ subject to Dirichlet condition. We introduce it by analogue with $\mathcal{H}_{\varepsilon,\eta}^D$ as associated with the form

$$\begin{aligned} \mathfrak{h}_0^D(u, v) := & \sum_{i,j=1}^2 \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_0)} + \sum_{j=1}^2 \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_0)} \\ & + \sum_{j=1}^2 \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_0)} + (A_0 u, v)_{L_2(\Omega_0)} \end{aligned} \tag{1.3}$$

in $L_2(\Omega_0)$ with the domain $\mathfrak{D}(\mathfrak{h}_0^D) := W_{2,0}^1(\Omega_0, \partial\Omega_0)$. The domain of operator \mathcal{H}_0^D is $W_{2,0}^2(\Omega_0, \partial\Omega_0)$ that can be shown by analogy with [44, Ch. III, Sec. 7,8], [45, Lm. 2.2].

Our first main result describes the uniform resolvent convergence for $\mathcal{H}_{\varepsilon,\eta}^D$.

Theorem 1.1. *Let $f \in L_2(\Omega_0)$. For sufficiently small ε the estimate*

$$\|(\mathcal{H}_{\varepsilon,\eta}^D - i)^{-1} f - (\mathcal{H}_0^D - i)^{-1} f\|_{W_2^1(\Omega_\varepsilon)} \leq C \eta^{1/2} \|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f .

The next four theorems describe the resolvent convergence for operator $\mathcal{H}_{\varepsilon,\eta}^R$. Given $a_0 \in W_\infty^1(\Gamma_0)$, let \mathcal{H}_0^R be the self-adjoint operator in $L_2(\Omega_0)$ associated with the lower-semibounded sesquilinear symmetric form

$$\begin{aligned} \mathfrak{h}_0^R(u, v) := & \sum_{i,j=1}^2 \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_0)} + \sum_{j=1}^2 \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_0)} \\ & + \sum_{j=1}^2 \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_0)} + (A_0 u, v)_{L_2(\Omega_0)} + (a_0 u, v)_{L_2(\Gamma_0)} \end{aligned}$$

with the domain $\mathfrak{D}(\mathfrak{h}_0^R) := W_{2,0}^1(\Omega_0, \Gamma)$. It can be shown by analogy with [44, Ch. III, Sec. 7,8], [45, Lm. 2.2] that the domain of \mathcal{H}_0^R consists of the functions $u \in W_{2,0}^2(\Omega_0, \Gamma)$ satisfying Robin condition

$$\left(\frac{\partial}{\partial \nu^0} + a_0 \right) u = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial}{\partial \nu^0} := - \sum_{i=1}^2 A_{i2} \frac{\partial}{\partial x_i} - \overline{A}_2. \quad (1.4)$$

First we consider the particular case of Neumann condition on Γ_ε , i.e., $a = 0$. Operator $\mathcal{H}_{\varepsilon,\eta}^R$ and associated quadratic form $\mathfrak{h}_{\varepsilon,\eta}^R$ are re-denoted in this case by $\mathcal{H}_{\varepsilon,\eta}^N$ and $\mathfrak{h}_{\varepsilon,\eta}^N$. By \mathcal{H}_0^N we denote the self-adjoint lower-semibounded operator in $L_2(\Omega_0)$ associated with the sesquilinear form \mathfrak{h}_0^N which is \mathfrak{h}_0^R taken for $a_0 \equiv 0$. Its domain is the set of the functions in $W_{2,0}^2(\Omega_0, \Gamma)$ satisfying boundary condition (1.4) with $a_0 = 0$. The resolvent convergence in this case is given in

Theorem 1.2. *Let $f \in L_2(\Omega_\varepsilon)$. Then for sufficiently small ε the estimate*

$$\|(\mathcal{H}_{\varepsilon,\eta}^N - i)^{-1} f - (\mathcal{H}_0^N - i)^{-1} f\|_{W_2^1(\Omega_\varepsilon)} \leq C \eta^{1/2} \|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f .

Assume now $a \neq 0$. Here we consider separately two cases,

$$\varepsilon^{-1} \eta(\varepsilon) \rightarrow \alpha = \text{const} \geq 0, \quad \varepsilon \rightarrow +0, \quad (1.5)$$

$$\varepsilon^{-1} \eta(\varepsilon) \rightarrow +\infty, \quad \varepsilon \rightarrow +0. \quad (1.6)$$

The first assumption means that the amplitude of the oscillation of curve Γ_ε is of the same order (or smaller) as the period. The other assumption corresponds to the case when the amplitude is much greater than the period. In what follows the first case is referred to as a relatively slow oscillating boundary Γ_ε while the other describes relatively high oscillating boundary Γ_ε .

We begin with the slowly oscillating boundary. We denote

$$a_0(x_1) := a(x_1, 0) \int_0^1 \sqrt{1 + \alpha^2 (b'(t))^2} dt. \quad (1.7)$$

Theorem 1.3. *Suppose (1.5) and let $f \in L_2(\Omega_\varepsilon)$. Then for sufficiently small ε the estimate*

$$\|(\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1} f - (\mathcal{H}_0^R - i)^{-1} f\|_{W_2^1(\Omega_\varepsilon)} \leq C(\eta^{1/2}(\varepsilon) + |\varepsilon^{-2} \eta^2(\varepsilon) - \alpha^2|) \|f\|_{L_2(\Omega_0)}$$

holds true, where function a_0 in (1.4) is defined in (1.7), and C is a constant independent of ε and f .

We proceed to the case of the highly oscillating boundary Γ_ε . Here the homogenized operator happens to be quite sensitive to the sign of a and zero level set of this function. In the paper we describe the resolvent convergence as a is non-negative. We first suppose that a is bounded from below by a positive constant. Surprisingly, but here the homogenized operator has the Dirichlet condition on Γ_0 as in Theorem 1.1.

Theorem 1.4. *Suppose (1.6),*

$$a(x) \geq c_1 > 0, \quad c_1 = \text{const}, \quad (1.8)$$

and that the function b is not identically constant. Let $f \in L_2(\Omega_0)$. Then for sufficiently small ε the estimate

$$\|(\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1}f - (\mathcal{H}_0^D - i)^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \leq C(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2})\|f\|_{L_2(\Omega_0)} \quad (1.9)$$

holds true, where C is a constant independent of ε and f .

In the next theorem we still suppose that a is non-negative but can have zeroes. An essential assumption is that zero level set of a is of zero measure. We let $b_* := \max_{[0,1]} b$.

Theorem 1.5. *Suppose (1.6),*

$$a \geq 0, \quad (1.10)$$

and that the function b is not identically constant. Assume also that for all sufficiently small δ the set $\{x : a(x) \leq \delta, 0 < x_2 < (b_ + 1)\eta\}$ is contained in an at most countable union of the rectangles $\{x : |x_1 - X_n| < \mu(\delta), 0 < x_2 < (b_* + 1)\eta\}$, where $\mu(\delta)$ is a some nonnegative function such that $\mu(\delta) \rightarrow +0$ as $\delta \rightarrow +0$, and numbers $X_n, n \in \mathbb{Z}$, are independent of δ , are taken in the ascending order, and satisfy uniform in n and m estimate*

$$|X_n - X_m| \geq c > 0, \quad n \neq m. \quad (1.11)$$

Let $f \in L_2(\Omega_0)$. Then for sufficiently small ε the estimate

$$\begin{aligned} & \|(\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1}f - (\mathcal{H}_0^D - i)^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \\ & \leq C(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2} + \mu^{1/2}(\delta)|\ln \mu(\delta)|^{1/2})\|f\|_{L_2(\Omega_0)} \end{aligned} \quad (1.12)$$

holds true, where C is a constant independent of ε and f , and $\delta = \delta(\varepsilon)$ is any function tending to zero as $\varepsilon \rightarrow +0$.

Let us discuss the main results. We first observe that under the hypotheses of all theorems we have the corresponding spectral convergence, namely, the convergence of the spectrum and the associated spectral projectors – see, for instance, [46, Thms. VIII.23, VIII.24]. We also stress that in all Theorems 1.1-1.5 the resolvent convergence is established in the sense of the uniform norm of bounded operator acting from $L_2(\Omega_0)$ into $W_2^1(\Omega_\varepsilon)$.

In the case of the Dirichlet condition on Γ_ε the homogenized operator has the same condition on Γ_0 no matter how the boundary Γ_ε oscillates, slowly or highly. The estimate for the rate of convergence is also universal being $\mathcal{O}(\eta^{1/2})$. Despite here we consider a periodically oscillating boundary, in the proof of Theorem 1.1 this fact is not used. This is why its statement is valid also for a periodically oscillating boundary described by the equation $x_2 = \eta b(x_1, \varepsilon)$, where b is an arbitrary function bounded uniformly in ε and such that $b(\cdot, \varepsilon) \in C(\mathbb{R})$. The estimate in Theorem 1.1 is sharp, see the discussion in the end of Sec. 2.

Similar situation happens if we have Neumann condition on Γ_ε . Here Theorem 1.2 says that the homogenized operator is subject to Neumann condition on Γ_0 and the rate of the uniform resolvent convergence is the same as in Theorem 1.1, namely, $\mathcal{O}(\varepsilon^{1/2})$. This estimate is again sharp, as the example in the end of Sec. 3 shows.

Once we have Robin condition on Γ_ε , the situation is completely different. If the boundary oscillates slowly, the homogenized operator still has Robin condition on Γ_0 , but the coefficient depends on the geometry of the original oscillations, cf. (1.7). The estimate for the rate of the resolvent convergence in this case involves additional term in comparison with the Dirichlet or Neumann case, cf. Theorem 1.3. The estimate in this theorem is again sharp, see the example in the end of Sec. 3.

As boundary Γ_ε oscillates relatively high, the resolvent convergence changes dramatically. If coefficient a is strictly positive, the homogenized operator has the Dirichlet

condition on Γ_0 . A new term, $\varepsilon^{1/2}\eta^{-1/2}$, appears in the estimate for the rate of the uniform resolvent convergence, cf. Theorem 1.4. We are able to prove that this term is sharp, see the discussion in the end of Sec. 4.

Provided function a is non-negative and vanishes only on a set of zero measure, the homogenized operator still has Dirichlet condition on Γ_0 , but the estimate for the rate of the uniform resolvent convergence becomes worse. Namely, the behavior of a in a vicinity of its zeroes becomes important. It is reflected by functions $\mu(\delta)$ and δ in (1.12). The latter should be chosen so that $\delta \rightarrow +0$, $\varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2} \rightarrow +0$, $\varepsilon \rightarrow +0$, that is always possible. The optimal choice of δ is so that

$$\begin{aligned}\mu^{1/2}(\delta)|\ln \mu(\delta)|^{1/2} &\sim \varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2}, \\ \delta\mu(\delta)|\ln \mu(\delta)| &\sim \varepsilon\eta^{-1}.\end{aligned}\tag{1.13}$$

As we see, the choice of δ depends on a particular structure of $\mu(\delta)$. The most typical case is $\mu(\delta) \sim \delta^{1/2}$, i.e., the function a vanishes by the quadratic law in a vicinity of its zeroes. In this case condition (1.13) becomes

$$\delta^{3/2}|\ln \delta| \sim \varepsilon\eta^{-1}$$

that implies

$$\delta \sim \varepsilon^{2/3}\eta^{-2/3}|\ln \varepsilon\eta^{-1}|^{-2/3}.$$

Then the estimate for the resolvent convergence in Theorem 1.5 is of order $\mathcal{O}((\eta^{1/2} + \varepsilon^{1/6}\eta^{-1/6}|\ln \varepsilon\eta^{-1}|^{1/3}))$.

We are not able to prove the sharpness of estimate (1.12), but in the end of Sec. 4 we provide some arguments showing that estimate (1.12) is rather close to be optimal.

In conclusion we discuss the case of Robin condition on highly oscillating Γ_ε when the coefficient a does not satisfy the hypotheses of Theorems 1.4, 1.5. If it is still non-negative but vanishes on a set of non-zero measure, and at the end-points of this set the vanishing happens with certain rate like in Theorem 1.5, we conjecture that the homogenized operator involves mixed Dirichlet and Neumann condition on Γ_0 . Namely, if $a(x_1, 0) \equiv 0$ on Γ_0^N and $a(x_1, 0) > 0$ on Γ_0^D , $\Gamma_0 = \Gamma_0^N \cup \Gamma_0^D$, it is natural to expect that the homogenized operator has Neumann condition on Γ_0^N and Dirichlet one on Γ_0^D . This conjecture can be regarded as the mixture of the statements of Theorems 1.2 and 1.5. The main difficulty of proving this conjecture is that the domain of such homogenized operator is no longer a subset of $W_2^2(\Omega_0)$ because of the mixed boundary conditions. At the same time, this fact was essentially used in all our proofs. Even a more complicated situation occurs once a is negative or sign-indefinite. If a is negative on a set of non-zero measure, it can be shown that the bottom of the spectrum of the perturbed operator goes to $-\infty$ as $\varepsilon \rightarrow +0$. In such case one should study the resolvent convergence near this bottom, i.e., for the spectral parameter tending to $-\infty$. This makes the issue quite troublesome. We stress that under the hypotheses of all Theorems 1.1-1.5 the bottom of the spectrum is lower-semibounded uniformly in ε .

2 Dirichlet condition

In this section we study the resolvent convergence of the operator $\mathcal{H}_{\varepsilon, \eta}^D$ and prove Theorem 1.1. We begin with auxiliary lemma.

Lemma 2.1. *Suppose $u \in W_{2,0}^2(\Omega_0, \Gamma_0)$, $v \in W_{2,0}^1(\Omega_\varepsilon, \Gamma_\varepsilon)$. Then the estimates*

$$\begin{aligned}|u(x)|^2 &\leq Cx_2^2\|u(x_1, \cdot)\|_{W_2^2(0,d)}^2, & \text{for a.e. } x_1 \in \mathbb{R}, \quad x_2 \in (0, d/2), \\ |\nabla u(x)|^2 &\leq C\|\nabla u(x_1, \cdot)\|_{W_2^1(0,d)}^2, & \text{for a.e. } x_1 \in \mathbb{R}, \quad x_2 \in (0, d/2), \\ |v(x)|^2 &\leq Cx_2\|v(x_1, \cdot)\|_{W_2^1(\eta b(x_1\varepsilon^{-1}), d)}^2, & \text{for a.e. } x_1 \in \mathbb{R}, \quad x_2 \in (\eta b(x_1\varepsilon^{-1}), d/2),\end{aligned}$$

hold true, where C are constants independent of x , ε , u , and v .

Proof. Since $u \in W_2^2(\Omega_0)$, for almost all $x_1 \in \mathbb{R}$ we have $u(x_1, \cdot) \in W_2^2(0, d)$. We represent the function u as

$$u(x_1, x_2) = \int_0^{x_2} \frac{\partial u}{\partial x_2}(x_1, t) dt,$$

and by Cauchy-Schwarz inequality we obtain

$$|u(x_1, x_2)|^2 \leq C x_2 \int_0^{x_2} \left| \frac{\partial u}{\partial x_2}(x_1, t) \right|^2 dt. \quad (2.1)$$

Let $\chi_1 = \chi_1(x_2)$ be an infinitely differentiable smooth function vanishing as $x_2 > 3d/4$ and equalling one as $x_2 < d/2$. Then for $x_2 \in [0, d/2]$ we have

$$\frac{\partial u}{\partial x_2}(x_1, x_2) = \int_d^{x_2} \left(\frac{\partial}{\partial x_2} \chi_1 \frac{\partial u}{\partial x_2} \right)(x_1, t) dt,$$

and thus

$$\left| \frac{\partial u}{\partial x_2}(x_1, x_2) \right|^2 \leq C \int_0^d \left(\left| \frac{\partial^2 u}{\partial x_2^2}(x_1, t) \right|^2 + \left| \frac{\partial u}{\partial x_2}(x_1, t) \right|^2 \right) dt.$$

Substituting this inequality into (2.1), we arrive at the first required estimate. To prove two others one should proceed as above starting with the representation

$$v(x_1, x_2) = \int_0^{x_2} \frac{\partial v}{\partial x_2}(x_1, t) dt,$$

where v is assumed to be extended by zero outside Ω_ε , and the representation

$$\frac{\partial u}{\partial x_j}(x_1, x_2) = \int_d^{x_2} \left(\frac{\partial}{\partial x_2} \chi_1 \frac{\partial u}{\partial x_j} \right)(x_1, t) dt.$$

□

Proof of Theorem 1.1. By $\chi_2 = \chi_2(t)$ we denote an infinitely differentiable non-negative cut-off function with the values in $[0, 1]$ vanishing as $t > 1$ and being one as $t < 0$. We also assume that the values of χ_2 are in $[0, 1]$. We choose a function K as

$$K(x_2, \eta) := \chi_2 \left(\frac{x_2 - b_* \eta}{\eta} \right). \quad (2.2)$$

We observe that the function $(1 - K)$ vanishes for $0 < x_2 < b_* \eta$ and is independent of x_1 .

Given a function $f \in L_2(\Omega_0)$, we denote $u_\varepsilon := (\mathcal{H}_{\varepsilon, \eta}^D - \mathbf{i})^{-1} f$, $u_0 := (\mathcal{H}_0^D - \mathbf{i})^{-1} f$, $v_\varepsilon := u_\varepsilon - (1 - K)u_0$. In accordance with the definition of u_ε and u_0 , these functions satisfy the integral identities

$$\mathfrak{h}_{\varepsilon, \eta}^D(u_\varepsilon, \phi) + \mathbf{i}(u_\varepsilon, \phi)_{L_2(\Omega_\varepsilon)} = (f, \phi)_{L_2(\Omega_\varepsilon)} \quad (2.3)$$

for each $\phi \in W_{2,0}^1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$, and

$$\mathfrak{h}_0^D(u_0, \phi) + \mathbf{i}(u_0, \phi)_{L_2(\Omega_0)} = (f, \phi)_{L_2(\Omega_0)} \quad (2.4)$$

for each $\phi \in W_{2,0}^1(\Omega_0, \partial\Omega_0)$. It is clear that $(1 - K)v_\varepsilon \in W_{2,0}^1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$, $(1 - K)u_0 \in W_{2,0}^1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$, and the extension of v_ε by zero in $\Omega_0 \setminus \Omega_\varepsilon$ belongs to $W_{2,0}^1(\Omega_0, \partial\Omega_0)$.

Bearing these facts in mind, as the test function in (2.3) we choose $\phi = v_\varepsilon$, and in (2.4) we let $\phi = (1 - K)v_\varepsilon$ assuming that v_ε is extended by zero in $\Omega_0 \setminus \Omega_\varepsilon$. It yields

$$\begin{aligned}\mathfrak{h}_{\varepsilon,\eta}^D(u_\varepsilon, v_\varepsilon) + i(u_\varepsilon, v_\varepsilon) &= (f, v_\varepsilon)_{L_2(\Omega_\varepsilon)}, \\ \mathfrak{h}_0^D(u_0, (1 - K)v_\varepsilon) + i(u_0, (1 - K)v_\varepsilon)_{L_2(\Omega_\varepsilon)} &= (f, (1 - K)v_\varepsilon)_{L_2(\Omega_\varepsilon)}.\end{aligned}$$

Employing (1.3), we rewrite the term $\mathfrak{h}_0^D(u_0, (1 - K)v_\varepsilon)$,

$$\begin{aligned}\mathfrak{h}_0^D(u_0, (1 - K)v_\varepsilon) &= \sum_{i,j=1}^2 \left((1 - K)A_{ij} \frac{\partial u_0}{\partial x_j}, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^2 \left(A_j(1 - K) \frac{\partial u_0}{\partial x_j}, v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \\ &\quad + \sum_{j=1}^2 \left((1 - K)u_0, A_j \frac{\partial v_\varepsilon}{\partial x_j} \right)_{L_2(\Omega_\varepsilon)} + (A_0(1 - K)u_0, v_\varepsilon)_{L_2(\Omega_\varepsilon)} \\ &\quad - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \\ &= \mathfrak{h}_{\varepsilon,\eta}^D((1 - K)u_0, v_\varepsilon) + \left(A_2 u_0, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \\ &\quad - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} + \sum_{i=1}^2 \left(A_{i2} u_0 \frac{\partial K}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)}.\end{aligned}\tag{2.5}$$

It implies

$$\begin{aligned}\mathfrak{h}_{\varepsilon,\eta}^D(v_\varepsilon, v_\varepsilon) + i\|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2 &= (f, K v_\varepsilon)_{L_2(\Omega_\varepsilon)} \\ &\quad + \left(A_2 u_0, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \\ &\quad - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} + \sum_{i=1}^2 \left(A_{i2} u_0 \frac{\partial K}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)}.\end{aligned}\tag{2.6}$$

The main idea of the proof is to estimate the right hand side of (2.6) with the introduced function K and get in this way an estimate for v_ε .

We first observe obvious inequalities

$$\|u_0\|_{W_2^1(\Omega_0)} \leq C\|f\|_{L_2(\Omega_0)}, \quad \|u_\varepsilon\|_{W_2^1(\Omega_0)} \leq C\|f\|_{L_2(\Omega_0)}.$$

Here and till the end of the section by C we denote inessential constants independent of ε , x , and f . Proceeding as in [44, Ch. III, Sec. 7,8] (see also [45, Lm. 2.2]), one can also check that

$$\|u_0\|_{W_2^2(\Omega_0)} \leq C\|f\|_{L_2(\Omega_0)}.\tag{2.7}$$

Denote $\Omega^\eta := \Omega_\varepsilon \cap \{x : 0 < x_2 < (b_* + 1)\eta\}$. Since the function K vanishes outside Ω^η and $|\nabla K| \leq C\eta^{-1}$, $0 \leq K \leq 1$, it is easy to see that

$$\begin{aligned}&\left| (f, K v_\varepsilon)_{L_2(\Omega_\varepsilon)} + \left(A_2 u_0, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \right. \\ &\quad \left. - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} + \sum_{i=1}^2 \left(A_{i2} u_0 \frac{\partial K}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} \right| \\ &\leq C \left(\|f\|_{L_2(\Omega_\varepsilon)} \|v_\varepsilon\|_{L_2(\Omega^\eta)} + \eta^{-1} \|u_0\|_{W_2^1(\Omega^\eta)} \|v_\varepsilon\|_{L_2(\Omega^\eta)} \right. \\ &\quad \left. + \eta^{-1} \|u_0\|_{L_2(\Omega^\eta)} \|\nabla v_\varepsilon\|_{L_2(\Omega^\varepsilon)} \right).\end{aligned}\tag{2.8}$$

We estimate the terms in the right hand side by applying Lemma 2.1,

$$\begin{aligned}
\|v_\varepsilon\|_{L_2(\Omega^\eta)}^2 &= \int_{\mathbb{R}} \|v_\varepsilon(x_1, \cdot)\|_{L_2(\eta b(x\varepsilon^{-1}), (b_*+1)\eta)}^2 dx_1 \\
&\leq C \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}^2 \int_0^{(b_*+1)\eta} x_2 dx_2 \leq C \eta^2 \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}^2, \\
\|u_0\|_{L_2(\Omega^\eta)}^2 &\leq C \|u_0\|_{W_2^2(\Omega_\varepsilon)}^2 \int_0^{(b_*+1)\eta} x_2^2 dx_2 \leq C \eta^3 \|u_0\|_{W_2^2(\Omega_\varepsilon)}^2, \\
\|\nabla u_0\|_{L_2(\Omega^\eta)}^2 &\leq C \|u_0\|_{W_2^2(\Omega_\varepsilon)}^2 \int_0^{(b_*+1)\eta} dx_2 \leq C \eta \|u_0\|_{W_2^2(\Omega_\varepsilon)}^2.
\end{aligned} \tag{2.9}$$

We substitute the obtained estimates and (2.7) into (2.8),

$$\begin{aligned}
&\left| (f, K v_\varepsilon)_{L_2(\Omega_\varepsilon)} + \left(A_2 u_0, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} \right. \\
&\quad \left. - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} + \sum_{i=1}^2 \left(A_{i2} u_0 \frac{\partial K}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} \right| \\
&\leq C \eta^{1/2} \|f\|_{L_2(\Omega_\varepsilon)} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.
\end{aligned}$$

We take the real and imaginary parts of the right hand side in (2.6) and employ then the last obtained estimate and (2.7). It leads us to the final estimate for v_ε ,

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leq C \eta^{1/2} \|f\|_{L_2(\Omega_\varepsilon)}.$$

Using (2.7), by analogy with (2.9) one can check easily that

$$\|K u_0\|_{W_2^1(\Omega_0)} \leq C \eta^{1/2} \|u_0\|_{W_2^2(\Omega_0)} \leq C \eta^{1/2} \|f\|_{L_2(\Omega_0)}. \tag{2.10}$$

The statement of Theorem 1.1 follows from two last estimates and the definition of v_ε . The proof is complete. \square

In conclusion let us discuss the optimality of the estimate in Theorem 1.1. Suppose for simplicity that the differential expression for $\mathcal{H}_{\varepsilon, \eta}^D$ is just a negative Laplacian, $\eta(\varepsilon) \equiv \varepsilon$, $b \in C^\infty[0, 1]$, $f \rightarrow C_0^\infty(\Omega_0)$, $u_0 \in C^\infty(\overline{\Omega_0})$ and u_0 vanishes for sufficiently large $|x_2|$. Under such assumptions by the method of matching of asymptotic expansions [47] and the multiscale method [48] one can construct the asymptotic expansion for u_ε ; for a similar spectral problem see [5], [8]. The asymptotics holds in $W_2^1(\Omega_\varepsilon)$ -norm and for $\varepsilon b(x_1 \varepsilon^{-1}) < x_2 < \varepsilon^{1/2}$ it reads as

$$u_\varepsilon(x) = \frac{\partial u_0}{\partial x_2}(x_1, 0)(x_2 + \varepsilon Y(x\varepsilon^{-1})) + \mathcal{O}(\varepsilon),$$

where $Y = Y(\xi)$, $\xi = (\xi_1, \xi_2)$ is the 1-periodic solution to the boundary value problem

$$\begin{aligned}
\Delta_\xi Y &= 0, & \xi_1 &\in (0, 1), & \xi_2 &> b(\xi_1), \\
Y &= -\xi_2, & \xi_1 &\in (0, 1), & \xi_2 &= b(\xi_1),
\end{aligned}$$

decaying exponentially as $\xi_2 \rightarrow +\infty$. Expanding then u_0 into Taylor series as $x_2 \rightarrow +0$,

one can check easily that

$$\begin{aligned}
\|u_\varepsilon - u_0\|_{W_2^1(\Omega_\varepsilon \cap \{x: x_2 < \varepsilon^{1/2}\})} &= \varepsilon \left\| \nabla_x \frac{\partial u_0}{\partial x_2}(x_1, 0) Y(x\varepsilon^{-1}) \right\|_{L_2(\Omega_\varepsilon \cap \{x: x_2 < \varepsilon^{1/2}\})} + \mathcal{O}(\varepsilon) \\
&= \left\| \frac{\partial u_0}{\partial x_2}(x_1, 0) \nabla_\xi Y(x\varepsilon^{-1}) \right\|_{L_2(\Omega_\varepsilon \cap \{x: x_2 < \varepsilon^{1/2}\})} + \mathcal{O}(\varepsilon) \\
&= \varepsilon^{1/2} \left(\int_{\mathbb{R}} dx_1 \left| \frac{\partial u_0}{\partial x_2}(x_1, 0) \right|^2 \int_{b(x_1\varepsilon^{-1})}^{\varepsilon^{-1/2}} |\nabla_\xi Y(x_1\varepsilon^{-1}, \xi_2)|^2 d\xi_2 \right)^{1/2} + \mathcal{O}(\varepsilon) \\
&= \varepsilon^{1/2} \left(\int_{\mathbb{R}} dx_1 \left| \frac{\partial u_0}{\partial x_2}(x_1, 0) \right|^2 \int_{b(x_1\varepsilon^{-1})}^{+\infty} |\nabla_\xi Y(x_1\varepsilon^{-1}, \xi_2)|^2 d\xi_2 \right)^{1/2} + \mathcal{O}(\varepsilon)
\end{aligned}$$

and thus $\|u_\varepsilon - u_0\|_{W_2^1(\Omega_\varepsilon)} \geq C\varepsilon^{1/2}$. It proves the optimality of the estimate in Theorem 1.1.

3 Robin condition on relatively slow oscillating boundary and Neumann condition

In this section we study the resolvent convergence for operators $\mathcal{H}_{\varepsilon, \eta}^N$, $\mathcal{H}_{\varepsilon, \eta}^R$ and prove Theorems 1.2, 1.3. Throughout the section by C we indicate various inessential constants independent of ε , x , and f .

We begin with two auxiliary lemmata. In these lemmata and their proofs constants C are supposed to be independent of ε , x , and u .

The first lemma is an analogue of Lemma 2.1.

Lemma 3.1. *For all $u \in W_2^1(\Omega_\varepsilon)$ and almost each $x_1 \in \mathbb{R}$, $x_2 \in (\eta b(x_1\varepsilon^{-1}), d/2)$ the estimate*

$$|u(x)| \leq C \|u(x_1, \cdot)\|_{W_2^1(0, \eta b(x_1\varepsilon^{-1}))},$$

holds true.

The proof of this lemma is similar to that of Lemma 2.1. One just should employ the obvious identity

$$u(x) = \int_d^{x_2} \frac{\partial \chi_1 u}{\partial x_2}(x_1, t) dt,$$

where χ_1 was defined in the proof of Lemma 2.1.

The next lemma gives an a priori estimate for the forms $\mathfrak{h}_{\varepsilon, \eta}^N$, $\mathfrak{h}_{\varepsilon, \eta}^R$.

Lemma 3.2. *For any $u \in W_{2,0}^1(\Omega_\varepsilon, \Gamma)$ the estimate*

$$\|u\|_{W_2^1(\Omega_\varepsilon)}^2 \leq C(\mathfrak{h}_{\varepsilon, \eta}^N(u, u) + \|u\|_{L_2(\Omega_\varepsilon)}^2) \quad (3.1)$$

holds true.

Suppose (1.5). Then for any $u \in W_{2,0}^1(\Omega_\varepsilon, \Gamma)$ the estimate

$$\|u\|_{W_2^1(\Omega_\varepsilon)}^2 \leq C(\mathfrak{h}_{\varepsilon, \eta}^R(u, u) + \|u\|_{L_2(\Omega_\varepsilon)}^2) \quad (3.2)$$

is valid.

Proof. It is clear that

$$\left| \sum_{j=1}^2 \left(A_j \frac{\partial u}{\partial x_j}, u \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^2 \left(u, A_j \frac{\partial u}{\partial x_j} \right)_{L_2(\Omega_\varepsilon)} \right| \leq \frac{c_0}{4} \|\nabla u\|_{L_2(\Omega_\varepsilon)}^2 + C \|u\|_{L_2(\Omega_\varepsilon)}^2. \quad (3.3)$$

This inequality and (1.1) imply (3.1).

To prove (3.2), we just need to estimate the boundary integral over Γ_ε in the definition of $\mathfrak{h}_{\varepsilon,\eta}^R$. For $x \in \Gamma_\varepsilon$ we have

$$|u(x)|^2 \leq \int_{\eta b(x_1 \varepsilon^{-1})}^d \frac{\partial |u|^2}{\partial x_2}(x_1, t) dt \leq \delta \|\nabla u\|_{L_2(\Omega_\varepsilon)}^2 + C(\delta) \|u\|_{L_2(\Omega_\varepsilon)}^2,$$

where the constant δ can be chosen arbitrarily small. Hence, due to (1.5), for an appropriate choice of δ

$$\begin{aligned} |(au, u)_{L_2(\Gamma_\varepsilon)}| &= \left| \int_{\mathbb{R}} a(x) |u(x)|^2 \sqrt{1 + \varepsilon^{-2} \eta^2 (b'(x_1 \varepsilon^{-1}))^2} dx_1 \right| \\ &\leq \frac{c_0}{4} \|\nabla u\|_{L_2(\Omega_\varepsilon)}^2 + C \|u\|_{L_2(\Omega_\varepsilon)}^2. \end{aligned}$$

By this inequality, (3.3), and (1.1) we get the desired estimate. \square

Proof of Theorem 1.2. Denote $u_\varepsilon := (\mathcal{H}_{\varepsilon,\eta}^N - i)^{-1} f$, $u_0 := (\mathcal{H}_0^N - i)^{-1} f$, $v_\varepsilon := u_\varepsilon - u_0$. The latter function solves the boundary value problem

$$\begin{aligned} \left(- \sum_{i,j=1}^2 \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum_{j=1}^2 A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 - i \right) v_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon, \\ v_\varepsilon &= 0 \quad \text{on } \Gamma, \quad \frac{\partial v_\varepsilon}{\partial \nu^\varepsilon} = - \frac{\partial u_0}{\partial \nu^\varepsilon} \quad \text{on } \Gamma_\varepsilon. \end{aligned}$$

The associated integral identity with v_ε taken as the test function is

$$\mathfrak{h}_{\varepsilon,\eta}^N(v_\varepsilon, v_\varepsilon) - i(v_\varepsilon, v_\varepsilon)_{L_2(\Omega_\varepsilon)} = - \left(\frac{\partial u_0}{\partial \nu^\varepsilon}, v_\varepsilon \right)_{L_2(\Gamma_\varepsilon)}. \quad (3.4)$$

Since

$$\nu^\varepsilon = \frac{1}{\sqrt{1 + \varepsilon^{-2} \eta^2 (\varepsilon) (b'(x_1 \varepsilon^{-1}))^2}} (-\varepsilon^{-1} \eta(\varepsilon) b'(x_1 \varepsilon^{-1}), 1),$$

we have

$$\left(\frac{\partial u_0}{\partial \nu^\varepsilon}, v_\varepsilon \right)_{L_2(\Gamma_\varepsilon)} = \int_{\mathbb{R}} (\varepsilon^{-1} \eta b'(x_1 \varepsilon^{-1}) w_1(x) - w_2(x)) \overline{v_\varepsilon}(x) \Big|_{x_2 = \eta b(x_1 \varepsilon^{-1})} dx_1, \quad (3.5)$$

$$w_j^\varepsilon := \sum_{i=1}^2 A_{ij} \frac{\partial u_0}{\partial x_i} + \overline{A_j} u_0.$$

Denote

$$w_3(x) := \int_{b_* \eta}^{x_2} w_1(x_1, t) \overline{v_\varepsilon}(x_1, t) dt,$$

where, we recall, $b_* := \max_{[0,1]} b$. The identity

$$\varepsilon^{-1} \eta w_1(x_1, \eta b(x_1 \varepsilon^{-1})) b'(x_1 \varepsilon^{-1}) = \frac{d}{dx_1} w_3(x_1, \eta b(x_1 \varepsilon^{-1})) - \frac{\partial w_3}{\partial x_1}(x_1, \eta b(x_1 \varepsilon^{-1})),$$

implies

$$\begin{aligned}
& \left| \varepsilon^{-1} \eta \int_{\mathbb{R}} b'(x_1 \varepsilon^{-1}) w_1 \bar{v}_\varepsilon \Big|_{x_2 = \eta b(x_1 \varepsilon^{-1})} dx_1 \right| \\
&= \left| \int_{\mathbb{R}} \frac{\partial w_3}{\partial x_1} \Big|_{x_2 = \eta b(x_1 \varepsilon^{-1})} dx_1 \right| = \left| \int_{\mathbb{R}} dx_1 \int_{\eta b_*}^{\eta b(x_1 \varepsilon^{-1})} \frac{\partial w_1^\varepsilon}{\partial x_1}(x) \bar{v}_\varepsilon dx_2 \right| \\
&\leq C \left(\|u_0\|_{W_2^2(\Omega_\varepsilon)} \|v_\varepsilon\|_{L_2(\Omega_\varepsilon \cap \{x: x_2 < b_* \eta\})} + \|u_0\|_{W_2^1(\Omega_\varepsilon) \cap \{x: x_2 < b_* \eta\}} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \right).
\end{aligned}$$

Applying Lemma 3.1 with $u = u_0$, $u = \frac{\partial u_0}{\partial x_i}$, $u = v_\varepsilon$, we obtain

$$\begin{aligned}
\|v_\varepsilon\|_{L_2(\Omega_\varepsilon \cap \{x: x_2 < \eta b_*\})} &\leq C \eta^{1/2} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}, \\
\|u_0\|_{W_2^1(\Omega_\varepsilon) \cap \{x: x_2 < \eta b_*\}} &\leq C \eta^{1/2} \|u_0\|_{W_2^2(\Omega_0)}.
\end{aligned}$$

Proceeding as [44, Ch. III, Sec. 7,8], [45, Lm. 2.2], one can estimate u_0 ,

$$\|u_0\|_{W_2^2(\Omega_0)} \leq C \|f\|_{L_2(\Omega_0)}. \quad (3.6)$$

Thus, by last three inequalities,

$$\left| \varepsilon^{-1} \eta \int_{\mathbb{R}} b'(x_1 \varepsilon^{-1}) w_1 \bar{v}_\varepsilon \Big|_{x_2 = \eta b(x_1 \varepsilon^{-1})} dx_1 \right| \leq C \eta^{1/2} \|f\|_{L_2(\Omega_0)} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}. \quad (3.7)$$

We estimate the second term in the right hand side of (3.5) as follows,

$$\left| \int_{\mathbb{R}} w_2(x) \bar{v}_\varepsilon(x) \Big|_{x_2 = \eta b(x_1 \varepsilon^{-1})} dx_1 \right| \leq \|w_2(\cdot, \eta b(\cdot \varepsilon^{-1}))\|_{L_2(\mathbb{R})} \|v_\varepsilon(\cdot, \eta b(\cdot \varepsilon^{-1}))\|_{L_2(\mathbb{R})}. \quad (3.8)$$

In view of the boundary condition for u_0 on Γ_0 , the function w_2 vanishes at $x_2 = 0$. Since it also belongs to $W_2^1(\Omega_0)$, by analogy with Lemma 2.1 one can prove easily that

$$\begin{aligned}
\|w_2(\cdot, \eta b(\cdot \varepsilon^{-1}))\|_{L_2(\mathbb{R})}^2 &\leq C \eta \|u_0(x_1, \cdot)\|_{W_2^2(0,d)}^2 \quad \text{for a.e. } x_1 \in \mathbb{R}, \\
\|w_2(\cdot, \eta b(\cdot \varepsilon^{-1}))\|_{L_2(\mathbb{R})} &\leq C \eta^{1/2} \|u_0\|_{W_2^2(\Omega_0)}.
\end{aligned}$$

The latter estimate, Lemma 3.1, (3.6), (3.5), (3.7), and (3.8) yield

$$\left| \left(\frac{\partial u_0}{\partial \nu^\varepsilon}, u_\varepsilon \right)_{L_2(\Gamma_\varepsilon)} \right| \leq C \eta^{1/2} \|f\|_{L_2(\Omega_\varepsilon)} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.$$

By Lemma 3.2 it implies

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leq C \eta^{1/2} \|f\|_{L_2(\Omega_\varepsilon)}.$$

The proof is complete. \square

Proof of Theorem 1.3. We follow the same lines as in the previous proof. Denote $u_\varepsilon := (\mathcal{H}_{\varepsilon, \eta}^R - i)^{-1} f$, $u_0 := (\mathcal{H}_0^R - i)^{-1} f$, $v_\varepsilon := u_\varepsilon - u_0$. By analogy with (3.4) we get

$$\mathfrak{h}_{\varepsilon, \eta}^N(v_\varepsilon, v_\varepsilon) - i(v_\varepsilon, v_\varepsilon)_{L_2(\Omega_\varepsilon)} = - \left(\left(\frac{\partial}{\partial \nu^\varepsilon} + a \right) u_0, v_\varepsilon \right)_{L_2(\Gamma_\varepsilon)}. \quad (3.9)$$

It follows from (3.5) that

$$\begin{aligned}
& \left(\left(\frac{\partial}{\partial \nu^\varepsilon} + a \right) u_0, v_\varepsilon \right)_{L_2(\Gamma_\varepsilon)} = \varepsilon^{-1} \eta \int_{\mathbb{R}} b'(x_1 \varepsilon^{-1}) w_1(x) \bar{v}_\varepsilon(x) \Big|_{x_2 = \eta b(x_1 \varepsilon^{-1})} dx_1 \\
& + \int_{\mathbb{R}} \left(a \sqrt{1 + \varepsilon^{-2} \eta^2 (b'(x_1 \varepsilon^{-1}))^2} u_0(x) - w_2(x) \right) \bar{v}_\varepsilon(x) \Big|_{x_2 = \eta b(x_1 \varepsilon^{-1})} dx_1,
\end{aligned} \quad (3.10)$$

and the first term in the right hand side can be again estimated by (3.7).

Due to the boundary condition on Γ_0 in operator \mathcal{H}_0^R we have $w_2 - a_0 u_0 = 0$ on Γ_0 and by analogy with Lemma 2.1 one can make sure that

$$\|w_2|_{x_2=\eta b(\cdot, \varepsilon^{-1})} - w_2|_{x_2=0}\|_{L_2(\mathbb{R})} \leq C\eta^{1/2}\|u_0\|_{W_2^2(\Omega_0)}. \quad (3.11)$$

Hence, to estimate the second term in the right hand side of (3.9), it is sufficient to estimate

$$\int_{\mathbb{R}} \left(a(x_1, \eta b(x_1 \varepsilon^{-1})) \sqrt{1 + \varepsilon^{-2} \eta^2 (b'(x_1 \varepsilon^{-1}))^2} - a_0(x_1) \right) u_0(x_1, 0) \bar{v}_\varepsilon(x_1, \eta b(x_1 \varepsilon^{-1})) dx_1. \quad (3.12)$$

Considering separately the cases $\alpha = 0$ and $\alpha \neq 0$, it is easy to see that

$$\left| a(x_1, \eta b(x_1 \varepsilon^{-1})) \sqrt{1 + \varepsilon^{-2} \eta^2 (b'(x_1 \varepsilon^{-1}))^2} - a_0(x_1) \right| \leq C(\eta^{1/2} + |\varepsilon^{-2} \eta^2 - \alpha^2|).$$

Thus, by Lemma 3.1 and (3.6), integral (3.12) can be estimated from above by $C(\eta^{1/2} + |\varepsilon^{-2} \eta^2 - \alpha^2|) \|f\|_{L_2(\Omega_0)} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}$. Together with (3.10), (3.7), (3.11) it leads us to the final estimate

$$\left| \left(\left(\frac{\partial}{\partial \nu^\varepsilon} + a \right) u_0, v_\varepsilon \right)_{L_2(\Gamma_\varepsilon)} \right| \leq C(\eta^{1/2} + |\varepsilon^{-2} \eta^2 - \alpha^2|) \|f\|_{L_2(\Omega_0)} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.$$

Substituting this estimate into (3.9) and applying Lemma 3.2, we complete the proof. \square

Let us show that the estimates in the proven theorems are sharp. Assume the differential expression for $\mathcal{H}_{\varepsilon, \eta}^R$ is the negative Laplacian, $2 \leq b(t) \leq 3$, the function a is constant and a_0 is determined by (1.7) via a . Let

$$F = F(x_2) = \begin{cases} 0, & \eta < x_2 < d, \\ 1, & 0 < x_2 < \eta, \end{cases}$$

and $U = U(x_2)$ be the solution to the boundary value problem

$$-U'' - iU = -iF \quad \text{in } (0, d), \quad U'(0) - a_0 U(0) = 0, \quad U(d) = 0.$$

The function U can be found explicitly,

$$U(x_2) = -\frac{k \sin k\eta + a_0(1 - \cos k\eta)}{k \cos kd + a_0 \sin kd} \sin k(d - x_2),$$

as $\eta < x_2 < d$, and

$$U(x_2) = 1 + \frac{a_0}{k} \sin kx_2 - \frac{k \cos k(d - \eta) + a_0 \sin kd}{k \cos kd + a_0 \sin kd} \left(\cos kx_2 + \frac{a_0}{k} \sin kx_2 \right),$$

as $0 < x_2 < \eta$, where $k := e^{i\pi/4}$. By $\varphi = \varphi(x_1)$ we denote an arbitrary function in $C_0^\infty(R)$ normalized in $L_2(\mathbb{R})$, and let $u_0(x) := \varphi(\eta x_1) U(x_2)$. The latter function satisfies $u_0 = (\mathcal{H}_0^R - i)^{-1} f$, where

$$f = (-\Delta - i)u_0 = f_1 + f_2, \quad f_1 := -i\varphi(\eta x_1)F(x_2), \quad f_2 := -\eta^2 \varphi''(\eta x_1)U(x_2).$$

It is straightforward to check that

$$\begin{aligned} \|f_1\|_{L_2(\Omega_0)} &= 1, \quad \|f_2\|_{L_2(\{x: \eta < x_2 < d\})} \leq C\eta^{5/2}, \\ \|u_0\|_{L_2(\{x: 3\eta < x_2 < d\})} &\geq C\eta^{1/2} \|f\|_{L_2(\Omega_0)}, \\ \|\nabla u_0\|_{L_2(\{x: 3\eta < x_2 < d\})} &\geq C\eta^{1/2} \|f\|_{L_2(\Omega_0)}. \end{aligned} \quad (3.13)$$

By Lemma 3.2 we have the a priori estimate

$$\|(\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \leq C\|f\|_{L_2(\Omega_\varepsilon)}$$

uniform in ε . Since $f = f_2$ on Ω_ε , by this estimate and (3.13) we get

$$\begin{aligned} \|(\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1}f\|_{W_2^1(\Omega_\varepsilon)} &\leq C\eta^{5/2}\|f\|_{L_2(\Omega_0)}, \\ \|(\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1}f - (\mathcal{H}_0^R - i)^{-1}f\|_{L_2(\Omega_\varepsilon)} &\geq C\eta^{1/2}\|f\|_{L_2(\Omega_0)}, \\ \|\nabla((\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1}f - (\mathcal{H}_0^R - i)^{-1}f)\|_{L_2(\Omega_\varepsilon)} &\geq C\eta^{1/2}\|f\|_{L_2(\Omega_0)}. \end{aligned}$$

Thus, as $a = 0$, the adduced example proves the sharpness of the estimate in Theorem 1.2. For arbitrary a it proves the sharpness of the term $\eta^{1/2}$ in the estimate in Theorem 1.3. The other term, $|\varepsilon^2\eta^{-2} - \alpha^2|$ is also sharp. Indeed, if we define the operator $\tilde{\mathcal{H}}_0^R$ in the same way as \mathcal{H}_0^R but replacing α^2 by $\varepsilon^2\eta^{-2}$ in (1.7), reproducing the proof of Theorem 1.3 we can make sure that

$$\|(\mathcal{H}_{\varepsilon,\eta}^R - i)^{-1}f - (\tilde{\mathcal{H}}_0^R - i)^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \leq C\eta^{1/2}\|f\|_{L_2(\Omega_0)}.$$

But then operator \mathcal{H}_0^R can be regarded as a regular perturbation of $\tilde{\mathcal{H}}_0^R$ and hence

$$\|(\mathcal{H}_0^R - i)^{-1}f - (\tilde{\mathcal{H}}_0^R - i)^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \sim |\varepsilon^2\eta^{-2} - \alpha^2|\|f\|_{L_2(\Omega_0)}.$$

Therefore, the estimate in Theorem 1.3 is sharp.

4 Robin condition on relatively high oscillating boundary

In this section we prove Theorems 1.4, 1.5. Throughout the proofs we indicate by C various inessential constants independent of ε , x , and f .

Proof of Theorem 1.4. Given a function $f \in L_2(\Omega_0)$, we let

$$u_\varepsilon := (\mathcal{H}_{\varepsilon,\eta}^D - i)^{-1}f, \quad u_0 := (\mathcal{H}_0^D - i)^{-1}f, \quad v_\varepsilon := u_\varepsilon - (1 - K)u_0,$$

where the function K is introduced by (2.2). We remind that the function $1 - K$ vanishes as $x_2 < b_*\eta$.

We write the integral identity for u_ε choosing v_ε as the test function,

$$\mathfrak{h}_{\varepsilon,\eta}^R(u_\varepsilon, v_\varepsilon) - i(u_\varepsilon, v_\varepsilon)_{L_2(\Omega_\varepsilon)} = (f, v_\varepsilon)_{L_2(\Omega_\varepsilon)}, \quad (4.1)$$

and that for u_0 with the test function $(1 - K)v_\varepsilon$ extended by zero in $\Omega_0 \setminus \Omega_\varepsilon$,

$$\mathfrak{h}_0^D(u_0, (1 - K)v_\varepsilon) - i(u_0, (1 - K)v_\varepsilon)_{L_2(\Omega_\varepsilon)} = (f, (1 - K)v_\varepsilon)_{L_2(\Omega_\varepsilon)}. \quad (4.2)$$

We observe that

$$(au, (1 - K)v)_{L_2(\Gamma_\varepsilon)} = 0$$

for all $u, v \in W_2^1(\Omega_\varepsilon)$. Bearing this fact in mind, we reproduce the arguments used in proving (2.5) and check easily that

$$\begin{aligned} \mathfrak{h}_0^D(u_0, (1 - K)v_\varepsilon) &= \mathfrak{h}_{\varepsilon,\eta}^R((1 - K)u_0, v_\varepsilon) + \left(A_2u_0, v_\varepsilon \frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)} - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon\right)_{L_2(\Omega_\varepsilon)} \\ &\quad - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)} + \sum_{i=1}^2 \left(A_{i2}u_0 \frac{\partial K}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}. \end{aligned}$$

We substitute this identity into (4.2) and calculate the difference of the result and (4.1),

$$\begin{aligned}
\mathfrak{h}_{\varepsilon,\eta}^R(v_\varepsilon, v_\varepsilon) - \|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2 &= (f, K v_\varepsilon)_{L_2(\Omega_\varepsilon)} + \left(A_2 u_0, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} \\
&\quad - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} \\
&\quad + \sum_{i=1}^2 \left(A_{i2} u_0 \frac{\partial K}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)}.
\end{aligned} \tag{4.3}$$

By the definition of K , (2.7), and Lemma 2.1 we have

$$\begin{aligned}
&\left| (f, K v_\varepsilon)_{L_2(\Omega_\varepsilon)} + \left(A_2 u_0, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} \right. \\
&\quad \left. - \left(u_0, A_2 \frac{\partial K}{\partial x_2} v_\varepsilon \right)_{L_2(\Omega_\varepsilon)} + \sum_{i=1}^2 \left(A_{i2} u_0 \frac{\partial K}{\partial x_2}, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} \right| \\
&\leq C \left(\|f\|_{L_2(\Omega^\eta)} \|v_\varepsilon\|_{L_2(\Omega^\eta)} + \eta^{-1} \|u_0\|_{L_2(\tilde{\Omega}^\eta)} \|v_\varepsilon\|_{W_2^1(\tilde{\Omega}^\eta)} \right) \\
&\leq C \eta^{1/2} \|f\|_{L_2(\Omega_0)} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)},
\end{aligned} \tag{4.4}$$

where, $\tilde{\Omega}^\eta := \{x : b_* \eta < x_2 < (b_* + 1)\eta\}$, and, we recall, $\Omega^\eta = \Omega_\varepsilon \cap \{x : 0 < x_2 < (b_* + 1)\eta\}$.

Denote

$$\tilde{b}(t) := \int_0^t |b'(z)| dz - t \int_0^1 |b'(z)| dz.$$

This function is continuous and 1-periodic. It satisfies the identity

$$\tilde{b}'(t) := |b'(t)| - \int_0^1 |b'(z)| dz.$$

Hence,

$$\begin{aligned}
&\int_0^1 |b'(t)| dt \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} = \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, |b'| \frac{\partial K}{\partial x_2} \bar{v}_\varepsilon \right)_{L_2(\tilde{\Omega}^\eta)} \\
&\quad - \varepsilon \sum_{j=1}^2 \int_{\Omega_\varepsilon} \tilde{b} \frac{\partial K}{\partial x_2} \frac{\partial}{\partial x_1} A_{2j} \frac{\partial u_0}{\partial x_j} \bar{v}_\varepsilon dx = - \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, |b'| \bar{v}_\varepsilon \right)_{L_2(\Gamma^\eta)} \\
&\quad - \sum_{j=1}^2 \int_{\tilde{\Omega}^\eta} K |b'| \frac{\partial}{\partial x_2} \left(A_{2j} \frac{\partial u_0}{\partial x_j} \bar{v}_\varepsilon \right) dx - \varepsilon \sum_{j=1}^2 \int_{\tilde{\Omega}^\eta} \tilde{b} \frac{\partial K}{\partial x_2} \frac{\partial}{\partial x_1} \left(A_{2j} \frac{\partial u_0}{\partial x_j} \bar{v}_\varepsilon \right) dx,
\end{aligned} \tag{4.5}$$

where $b' = b'(x_1 \varepsilon^{-1})$, $\tilde{b} = \tilde{b}(x_1 \varepsilon^{-1})$, $\Gamma^\eta := \{x : x_2 = b_* \eta\}$. Since $b(t)$ is not identically constant, in view of Lemma 2.1, and the definition of K we get

$$\begin{aligned}
&\left| \sum_{j=1}^2 \left(A_{2j} \frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial K}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} \right| \leq C \|f\|_{L_2(\Omega_0)} \| |b'|^{1/2} v_\varepsilon \|_{L_2(\Gamma^\eta)} \\
&\quad + C(\varepsilon \eta^{-1/2} + \eta^{1/2}) \|f\|_{L_2(\Omega_0)} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.
\end{aligned} \tag{4.6}$$

The identity

$$\begin{aligned} \| |b'|^{1/2} v_\varepsilon \|_{L_2(\Gamma^\eta)}^2 &= \int_{\mathbb{R}} dx_1 \int_{b_*\eta}^{\eta b(x_1\varepsilon^{-1})} |b'(x_1\varepsilon^{-1})| \frac{\partial |v_\varepsilon|^2}{\partial x_1} dx_2 + q_\varepsilon, \\ q_\varepsilon &:= \int_{\mathbb{R}} |b'(x_1\varepsilon^{-1})| |v_\varepsilon(x_1, \eta b(x_1\varepsilon^{-1}))|^2 dx_1, \end{aligned} \quad (4.7)$$

and Lemma 2.1 imply

$$\| |b'|^{1/2} v_\varepsilon \|_{L_2(\Gamma^\eta)} \leq C \eta^{1/2} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} + q_\varepsilon^{1/2}. \quad (4.8)$$

In view of the assumption (1.8) the boundary term in the definition of the form $\mathfrak{h}_{\varepsilon,\eta}^R$ can be estimated as

$$(av_\varepsilon, v_\varepsilon)_{L_2(\Gamma_\varepsilon)} \geq c_1 \int_{\mathbb{R}} |v_\varepsilon(x_1, \eta b(x_1\varepsilon^{-1}))|^2 \sqrt{1 + \varepsilon^{-2} \eta^2 (b'(x_1\varepsilon^{-1}))^2} dx_1 \geq \frac{c_1 \eta}{\varepsilon} q_\varepsilon. \quad (4.9)$$

This inequality, (3.1), (4.3), (4.4), (4.6), (4.8), (1.6) yield

$$\begin{aligned} \frac{\eta}{\varepsilon} q_\varepsilon + \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}^2 &\leq C \left(\eta^{1/2} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} + q_\varepsilon^{1/2} \right) \|f\|_{L_2(\Omega_0)} \\ &\leq C (\eta^{1/2} + \varepsilon^{1/2} \eta^{-1/2}) \left(\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}^2 + \frac{\eta}{\varepsilon} q_\varepsilon \right)^{1/2} \|f\|_{L_2(\Omega_0)}. \end{aligned}$$

It follows that

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}^2 \leq C (\eta^{1/2} + \varepsilon^{1/2} \eta^{-1/2}) \|f\|_{L_2(\Omega_0)}.$$

It remains to employ (2.10) to complete the proof. \square

Proof of Theorem 1.5. The proof of this theorem is similar to the previous one up to a substantial modification. All the arguments of the previous proof remain true up to inequality (4.8), while estimate (4.9) is no longer valid since we replace assumption (1.8) by (1.10). And the aforementioned modification is a new estimate substituting (4.9).

Given any $\delta > 0$, we split set Γ^η into two parts, $\Gamma^\eta = \Gamma_\delta^\eta \cup \Gamma^{\eta,\delta}$,

$$\Gamma_\delta^\eta := \{x : a(x_1, b_*\eta) > \delta, x_2 = b_*\eta\}, \quad \Gamma^{\eta,\delta} := \{x : a(x_1, b_*\eta) \leq \delta, x_2 = b_*\eta\},$$

and let $\gamma_\delta := \{x_1 \in \mathbb{R} : (x_1, b_*\eta) \in \Gamma_\delta^\eta\}$. We observe that

$$(av_\varepsilon, v_\varepsilon)_{L_2(\Gamma_\varepsilon)} \geq \frac{\eta}{\varepsilon} \int_{\gamma} a(x_1, \eta b(x_1\varepsilon^{-1})) |b'(x_1\varepsilon^{-1})| |v_\varepsilon(x_1, \eta b(x_1\varepsilon^{-1}))|^2 dx_1 := \frac{\eta}{\varepsilon} \mathfrak{q}_{\varepsilon,\delta}. \quad (4.10)$$

By analogy with (4.7), (4.8) we obtain

$$\| |b'|^{1/2} v_\varepsilon \|_{L_2(\Gamma_\delta^\eta)} \leq C \eta^{1/2} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} + \delta^{-1/2} \mathfrak{q}_{\varepsilon,\delta}^{1/2}. \quad (4.11)$$

The next auxiliary lemma will allow us to estimate $\|v_\varepsilon\|_{L_2(\Gamma^{\eta,\delta})}$.

Lemma 4.1. *Let $v \in W_2^1(\square)$, where $\square := \{x : d_- < x_1 < d_+, b_*\eta < x_2 < d\}$, $d_+ > d_-$, d_\pm are some constants. Denote $\Xi_\mu := \{x : |x_1 - d_0| < \mu, x_2 = b_*\eta\}$ and suppose that there exists a positive constant c independent of μ such that $d_0 - d_- \geq c$, $d_+ - d_0 \geq c$. Then for sufficiently small μ there exists a positive constant C independent of μ and v but dependent on c such that the inequality*

$$\|v\|_{L_2(\Xi_\mu)} \leq C \mu^{1/2} |\ln \mu|^{1/2} \|v\|_{W_2^1(\square)} \quad (4.12)$$

holds true.

Proof. We expand v in a Fourier series

$$v(x) = \sum_{m,n=0}^{\infty} c_{mn} \cos \frac{\pi n}{d_+ - d_-} (x_1 - d_-) \cos \frac{\pi m}{(d - \eta)} (x_2 - \eta) \quad (4.13)$$

converging in $W_2^1(\square)$. In view of the Parseval identity one has

$$\sum_{m,n=0}^{\infty} |c_{mn}|^2 (m^2 + n^2) \leq C \|v\|_{W_2^1(\square)}^2. \quad (4.14)$$

Due to the embedding of $W_2^1(\square)$ into $L_2(\Xi_\mu)$, we can employ (4.13) to calculate $\|v\|_{L_2(\Xi_\mu)}$,

$$\begin{aligned} \|v\|_{L_2(\Xi_\mu)}^2 &= \sum_{m,n,p,q=0}^{\infty} c_{mn} \overline{c_{pq}} \int_{d_0-\mu}^{d_0+\mu} \cos \frac{\pi n}{d_+ - d_-} (x_1 - d_-) \cos \frac{\pi p}{d_+ - d_-} (x_1 - d_-) dx_1 \\ &= \frac{d_+ - d_-}{\pi} \sum_{m,n,p,q=0}^{\infty} c_{mn} \overline{c_{pq}} \left(\sin \frac{\pi(n+p)\mu}{d_+ - d_-} \frac{\cos \pi(n+p)}{n+p} \right. \\ &\quad \left. - \sin \frac{\pi(n-p)\mu}{d_+ - d_-} \frac{\cos \pi(n-p)}{n-p} \right), \end{aligned}$$

where $\sin \frac{\pi(n-p)\mu}{d_+ - d_-} / (n-p)$ is to be replaced by $\pi\mu/d$ as $n = p$. We employ Cauchy-Schwarz inequality and the inequality

$$\sin^2 t \leq \frac{t^2}{1+t^2}, \quad t \geq 0,$$

and by (4.14) we obtain

$$\begin{aligned} \|v\|_{L_2(\Xi_\mu)}^4 &\leq \sum_{m,n,p,q=0}^{\infty} |c_{mn}|^2 |c_{pq}|^2 (m^2 + n^2)(p^2 + q^2) \\ &\quad \sum_{m,n,p,q=0}^{\infty} \frac{2}{(m^2 + n^2)(p^2 + q^2)} \left(\sin^2 \frac{\pi(n+p)\mu}{d_+ - d_-} \frac{1}{(n+p)^2} + \sin^2 \frac{\pi(n-p)\mu}{d_+ - d_-} \frac{1}{(n-p)^2} \right) \\ &\leq C\mu^2 \|v\|_{W_2^1(\square)}^4 \sum_{m,n,p,q=0}^{\infty} \frac{1}{(m^2 + n^2)(p^2 + q^2)} \left(\frac{1}{1 + \mu^2(n+p)^2} + \frac{1}{1 + \mu^2(n-p)^2} \right) \\ &\leq C\mu^2 \|v\|_{W_2^1(\square)}^4 \sum_{n,p=0}^{\infty} \left(\frac{1}{1 + \mu^2(n+p)^2} + \frac{1}{1 + \mu^2(n-p)^2} \right) \int_1^{+\infty} \frac{dz}{z^2 + n^2} \int_1^{+\infty} \frac{dz}{z^2 + p^2} \\ &\leq C\mu^2 \|v\|_{W_2^1(\square)}^4 \sum_{n,p=0}^{\infty} \frac{1}{1 + nq} \left(\frac{1}{1 + \mu^2(n+p)^2} + \frac{1}{1 + \mu^2(n-p)^2} \right). \end{aligned}$$

In the last sum we extract the terms for $(n, p) = (0, 0)$, $(n, p) = (0, 1)$, and $(n, p) = (1, 0)$. Then we replace the remaining summation by the integration and estimate in this way the sum by a two-dimensional integral,

$$\begin{aligned} \|v\|_{L_2(\Xi_\mu)}^4 &\leq 3\mu^2 \|v\|_{W_2^1(\square)}^4 \\ &\quad + C\mu^2 \|v\|_{W_2^1(\square)}^4 \int_{\substack{z_1^2 + z_2^2 > 3 \\ z_1, z_2 > 0}} \left(\frac{1}{1 + \mu^2(z_1 + z_2)^2} + \frac{1}{1 + \mu^2(z_1 - z_2)^2} \right) \frac{dz_1 dz_2}{1 + z_1 z_2}. \end{aligned}$$

Passing to the polar coordinates (r, θ) associated with (z_1, z_2) , we get

$$\begin{aligned}
& \int_{\substack{z_1^2 + z_2^2 > 3 \\ z_1, z_2 > 0}} \left(\frac{1}{1 + \mu^2(z_1 + z_2)^2} + \frac{1}{1 + \mu^2(z_1 - z_2)^2} \right) \frac{dz_1 dz_2}{1 + z_1 z_2} \\
& \leq 2 \int_{\sqrt{3}}^{+\infty} \int_0^{\pi/2} \left(\frac{1}{1 + \mu^2 r^2 (1 + \sin 2\theta)} + \frac{1}{1 + \mu^2 r^2 (1 - \sin 2\theta)} \right) \frac{r dr d\theta}{1 + r^2 \sin 2\theta} \\
& \leq C \int_3^{+\infty} \left(\frac{\ln \tau}{\tau(1 + \mu^2 \tau)} + \frac{\mu^2}{(1 + \mu^2 \tau)^{3/2}} \right) d\tau \\
& = C \int_{3\mu^2}^{+\infty} \left(\frac{\ln \tau - 2 \ln \mu}{\tau(1 + \tau)} + \frac{1}{(1 + \tau)^{3/2}} \right) d\tau \leq C \ln^2 \mu.
\end{aligned}$$

Two last formulas proves the desired estimate for $\|v\|_{L_2(\Xi_\mu)}$. \square

We apply the proven lemma with $v = v_\varepsilon$, $d_- = X_n - c/2$, $d_+ = X_n + c/2$, $d_0 = X_n$ and sum the obtained inequalities over $n \in \mathbb{Z}$. It gives the estimate for $\|v_\varepsilon\|_{L_2(\Gamma^{\eta, \delta})}$,

$$\|v_\varepsilon\|_{L_2(\Gamma^{\eta, \delta})} \leq C \mu^{1/2}(\delta) |\ln \mu(\delta)|^{1/2} \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.$$

This estimate and (4.11) imply

$$\begin{aligned}
\| |b'|^{1/2} v_\varepsilon \|_{L_2(\Gamma^\eta)} & \leq \| |b'|^{1/2} v_\varepsilon \|_{L_2(\Gamma_\delta^\eta)} + \| |b'|^{1/2} v_\varepsilon \|_{L_2(\Gamma^{\eta, \delta})} \\
& \leq C(\eta^{1/2} + \mu(\delta)^{1/2} |\ln \mu(\delta)|^{1/2}) \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} + \delta^{-1/2} \mathbf{q}_{\varepsilon, \delta}^{1/2}.
\end{aligned}$$

We substitute the obtained inequality and (4.4), (4.6), (4.10) into (4.3) and employ (3.1), (1.6). It results in

$$\begin{aligned}
\frac{\eta}{\varepsilon} \mathbf{q}_{\varepsilon, \delta} + \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} & \leq C(\eta^{1/2} + \mu^{1/2}(\delta) |\ln \mu(\delta)|^{1/2}) \|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \|f\|_{L_2(\Omega_0)} \\
& \quad + C \delta^{-1/2} \mathbf{q}_{\varepsilon, \delta}^{1/2} \|f\|_{L_2(\Omega_0)}
\end{aligned}$$

that leads us to the desired estimate

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leq C(\eta^{1/2} + \varepsilon^{1/2} \eta^{1/2} \delta^{-1/2} + \mu^{1/2}(\delta) |\ln \mu(\delta)|^{1/2}) \|f\|_{L_2(\Omega_0)}.$$

Together with (2.10) it completes the proof. \square

Let us discuss the sharpness of the estimates in the proven theorems. We begin with Theorem 1.4 and first show that the term $\varepsilon^{1/2} \eta^{-1/2}$ is sharp. In order to do it, we assume that $\eta^{1/2} \ll \varepsilon^{1/2} \eta^{-1/2}$, i.e., $\eta \varepsilon^{-1/2} \rightarrow +0$ as $\varepsilon \rightarrow +0$. We also suppose that the differential expression for operator $\mathcal{H}_{\varepsilon, \eta}^R$ is just the negative Laplacian. Then by (2.10)

$$\|u_\varepsilon - u_0 - v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leq C \eta^{1/2},$$

and it is sufficient to deal only with v_ε . Proceeding as in (4.3), (4.4), (4.5), one can check that v_ε can be represented as the sum $v_\varepsilon = v_\varepsilon^{(1)} + v_\varepsilon^{(2)}$, where $v_\varepsilon^{(i)} \in W_{2,0}^1(\Omega_\varepsilon, \Gamma)$ solve the integral identities,

$$\begin{aligned}
& (\nabla v_\varepsilon^{(1)}, \nabla \varphi)_{L_2(\Omega_\varepsilon)} - i(v_\varepsilon^{(1)}, \varphi)_{L_2(\Omega_\varepsilon)} + (a v_\varepsilon^{(1)}, \varphi)_{L_2(\Gamma_\varepsilon)} \\
& = (f, K \varphi)_{L_2(\Omega_\varepsilon)} + \left(u_0 \frac{\partial K}{\partial x_2}, \frac{\partial \varphi}{\partial x_2} \right)_{L_2(\Omega_\varepsilon)} \\
& \quad + \left(\int_0^1 |b'(t)| dt \right)^{-1} \left(\int_{\tilde{\Omega}^\eta} K |b'| \frac{\partial}{\partial x_2} \left(\frac{\partial u_0}{\partial x_2} \bar{\varphi}_\varepsilon \right) dx + \varepsilon \int_{\tilde{\Omega}^\eta} |\tilde{b}| \frac{\partial K}{\partial x_2} \frac{\partial}{\partial x_1} \left(\frac{\partial u_0}{\partial x_2} \bar{\varphi}_\varepsilon \right) dx \right),
\end{aligned}$$

and

$$\begin{aligned} & (\nabla v_\varepsilon^{(2)}, \nabla \varphi)_{L_2(\Omega_\varepsilon)} - i(v_\varepsilon^{(2)}, \varphi)_{L_2(\Omega_\varepsilon)} + (av^{(2)}, \varphi)_{L_2(\Gamma_\varepsilon)} \\ &= \left(\int_0^1 |b'(t)| dt \right)^{-1} \int_{\mathbb{R}} |b'(x_1 \varepsilon^{-1})| \frac{\partial u_0}{\partial x_2}(x_1, \eta b(x_1 \varepsilon^{-1})) \overline{\varphi}(x_1, \eta b(x_1 \varepsilon^{-1})) dx_1, \end{aligned} \quad (4.15)$$

where $\varphi \in W_{2,0}^1(\Omega_\varepsilon, \Gamma)$. As in (4.3), (4.4), (4.5), (4.7), one can show that

$$\|v_\varepsilon^{(1)}\|_{L_2(\Omega_\varepsilon)} \leq C\eta^{1/2}\|f\|_{L_2(\Omega_0)}$$

and it sufficient to show that $v_\varepsilon^{(2)}$ is indeed of order $\mathcal{O}(\varepsilon^{1/2}\eta^{-1/2})$. By Theorem 1.4 and the latter estimate,

$$\|v_\varepsilon^{(2)}\|_{L_2(\Omega_\varepsilon)} \leq C(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2})\|f\|_{L_2(\Omega_0)}, \quad (4.16)$$

and hence

$$\int_{\mathbb{R}} |v_\varepsilon^{(2)}(x_1, \eta b(x_1 \varepsilon^{-1}))|^2 dx_1 \leq C(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2})\|f\|_{L_2(\Omega_0)}. \quad (4.17)$$

In (4.15) we choose $\varphi(x) = \frac{\partial u_0}{\partial x_2}(x)\chi_1(x_2)$, where cut-off function χ_1 was introduced in the proof of Lemma 2.1. By (4.16) we then see that first two terms in the left hand side of (4.15) are estimated by $(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2})\|f\|_{L_2(\Omega_0)}^2$. It is clear that we can choose f so that $\|f\|_{L_2(\Omega_0)} = 1$ and

$$C_1 \leq \int_{\mathbb{R}} |b'(x_1 \varepsilon^{-1})| \left| \frac{\partial u_0}{\partial x_2}(x_1, \eta b(x_1 \varepsilon^{-1})) \right| dx_1 \leq C_2 \quad (4.18)$$

uniformly in ε . Hence, by (4.15) with $\varphi(x) = \frac{\partial u_0}{\partial x_2}(x)\chi_1(x_2)$, (4.16), (4.17),

$$\left| \int_{\mathbb{R}} |b'(x_1 \varepsilon^{-1})| v_\varepsilon^{(2)}(x_1, \eta b(x_1 \varepsilon^{-1})) \frac{\partial \overline{u_0}}{\partial x_2}(x_1, \eta b(x_1 \varepsilon^{-1})) dx_1 \right| \geq C\varepsilon\eta^{-1}.$$

In (4.15) we let now $\varphi = v_\varepsilon^{(2)}$ and get that $\|v_\varepsilon^{(2)}\|_{W_2^1(\Omega_\varepsilon)}$ is indeed of order $\mathcal{O}(\varepsilon^{1/2}\eta^{-1/2})$ and this order is sharp. Hence, the term $\varepsilon^{1/2}\eta^{-1/2}$ in estimate (1.9) is sharp.

To prove the sharpness of the other term, $\eta^{1/2}$, one needs to adduce some example, but we failed trying to find it. Nevertheless, we know that such term is sharp under the hypotheses of Theorems 1.1, 1.2, 1.3. In Theorem 1.4 the situation is more complicated since we have the oscillation is relatively high and we have Robin condition on the oscillating boundary. This is why the presence of the term $\eta^{1/2}$ in (1.9) is reasonable and it seems to be sharp. At least in the framework of the technique we employed, this estimate can not be improved since all the inequalities in the proof are sharp. We also note that similar estimate for the rate of the strong resolvent convergence in $L_2(\Omega_\varepsilon)$ -norm (not the uniform one!) established in [14] is worse than (1.9).

Estimate (1.12) is worse than (1.9) since we replace assumption (1.8) by (1.10). In this situation it is natural to have function μ involved in (1.12). Here we can not adduce an example proving the sharpness of this estimate. On the other hand, we still have the term $\eta^{1/2}$. The term $\varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2}$ is similar to $\varepsilon^{1/2}\eta^{-1/2}$ in (1.9). The presence of the factor $\delta^{-1/2}$ shows how conditions (1.10), (1.11) spoil the estimate in comparison with (1.9). The last term, $\mu(\delta)|\ln \mu(\delta)|$, also reflects the influence of the zeroes of a . It comes directly from (4.12) which is a sharp inequality. To prove the latter fact, it is sufficient to make sure that for

$$v(x) := \begin{cases} 1, & |x - (d_0, b_*\eta)| < 3\mu, \\ \frac{\ln |x - (d_0, b_*\eta)|}{\ln 3\mu}, & |x - (d_0, b_*\eta)| > 3\mu, \end{cases}$$

one has

$$\|v\|_{L_2(\Xi_\mu)} \geq C\mu^{1/2} |\ln \mu|^{1/2} \|v\|_{W_2^1(\square)}.$$

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